ON INFINITE SYSTEMS OF LINEAR INTEGRAL EQUAT ONS.

By Louis Brand.

The principal problem of this paper is the solution of a denumerably infinite system of linear integral equations of the type

$$\int_a^b \varphi_i(x)\xi(x)dx = b_i \qquad (i = 1, 2, \cdots),$$

the φ's being given functions and the b's given constants. The solution of such a system of equations, besides its intrinsic importance, enables us to approach in a simple manner other matters such as the existence of biorthogonal systems of functions and the expansion of an arbitrary function in terms of a given system of functions by the method of least squares. The method of procedure is closely analogous to that employed in an earlier paper in these *Annals** by Professor Bôcher and myself, treating of the solution of an infinite number of linear equations in an infinite number of unknowns, and is chiefly characterized by the preliminary solution of a finite number of equations and the treatment of the infinite system as a limiting case. The trend of this analogy is fully exhibited in Theorems 1 and 2; subsequently the proofs of the theorems having close analogues in (I) are given in much less detail—sometimes in mere outline when reference to this article will make the detailed procedure evident.

This paper is a development of a manuscript handed me by Professor Bôcher a year ago in which the method of proceeding for §§ 2, 3, and the beginning of § 6, was sketched in outline. This manuscript was written in June, 1910, several months before the appearance of a paper by F. Riesz† in which it was mentioned how Schmidt's method,‡ of which a modification is used in (I), can be applied to the problem here considered. The method here developed has in comparison to that suggested by Riesz§ all the advantages which the treatment in (I) has over the original treatment of that problem by Schmidt.

1. We will consider real or complex functions of the real variable x on the arbitrary interval $a \leqq x \leqq b$, denoting such functions by f, φ, etc.,

* June, 1912. This article will be referred to as (I).

† "Systeme integrierbare Funktionen," Mathematische Annalen, 69 (1910), p. 451, p. 469.

‡ "Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten," Rend. Circ. Matem. Palermo, 25 (1908), pp. 53–77.

§ The method actually used by Riesz in the article above cited differs from both of these and leads to different results.

and their conjugate imaginary functions by $\bar{f}$, $\bar{\varphi}$, etc., omitting the arguments. All integrals shall be taken in the sense of Lebesgue, and the term *integrable* shall mean integrable in this sense* on the interval (a, b). Ω is the class of all such functions φ for which $|\varphi|^2$ is integrable. It may be readily shown that if $|\varphi|^2$ is integrable (we denote functions of the class Ω by Greek letters) both $|\varphi|$ and φ are integrable.† But the chief characteristic of functions of the class Ω is expressed in the following theorem: *The product of any two functions of the class Ω is integrable; and every function which yields an integrable product with all the functions of the class Ω is itself a member of this class.*‡ From this theorem it follows that the sum of two functions of the class Ω is likewise a member of this class; or, more generally, any linear combination of functions of the class Ω belongs to this class.

A function is said to be *essentially zero* if it differs from zero merely on a point set of content§ zero. The integral of a function essentially zero is, of course, zero. The n functions $f_1, \cdots, f_n$ are said to be *essentially linearly dependent* in (a, b) if there exist n constants, $c_1, \cdots, c_n$, not all zero, such that $c_1f_1 + \cdots + c_nf_n$ is essentially zero in this interval.

By the *norm* of a function φ of the class Ω is understood the real constant

$$\text{norm } \varphi = \int_a^b \varphi\bar{\varphi}dx = \int_a^b |\varphi|^2 dx.$$

Norm $\varphi = 0$ when and only when φ is essentially zero, and is otherwise positive.

In the following we shall omit the limits of integration, a, b, and the dx in writing integrals; these are to be understood in all cases.

2. Consider now the system of n homogeneous integral equations

$$(1) \qquad \int \varphi_1\xi = 0, \qquad \int \varphi_2\xi = 0, \qquad \cdots, \qquad \int \varphi_n\xi = 0,$$

where $\varphi_1, \varphi_2, \cdots, \varphi_n$ are functions of the class Ω. We propose to determine all the functions ξ of the class Ω satisfying (1).

THEOREM 1. *If ξ is a solution of* (1) *which is essentially linearly dependent upon $\bar{\varphi}_1, \bar{\varphi}_2, \cdots, \bar{\varphi}_n$, then ξ is essentially zero.*

Suppose that

$$\xi = c_1\bar{\varphi}_1 + c_2\bar{\varphi}_2 + \cdots + c_n\bar{\varphi}_n$$

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* Lebesgue, Leçons sur l'intégration, Paris (1904), p. 115. Lebesgue terms functions integrable in this sense *sommable*. See also F. Riesz, l. c., § 1.

† Cf. F. Riesz, l. c., § 2.

‡ Lebesgue, "Sur les intégrales singulières," Annales de la Faculté des Sciences de Toulouse, 3° serie, t. I, p. 37, p. 39; F. Riesz, l. c., p. 459. Riesz proves a more general theorem than the one stated above.

§ Content as defined by Lebesgue, Leçons sur l'intégration, p. 102 et seq., i. e., *mesure*.

(essentially). Multiplying equations (1) by $\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_n$ respectively and adding we get

$$\int (\bar{c}_1\varphi_1 + \bar{c}_2\varphi_2 + \cdots + \bar{c}_n\varphi_n)\xi = \int \xi\bar{\xi} = \int |\xi|^2 = 0.$$

Hence ξ is essentially zero, as was to be proved.

We are now in position to obtain a criterion for the essential linear dependence of n functions of the class Ω. If $\varphi_1, \varphi_2, \cdots, \varphi_n$ are essentially linearly dependent

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n = 0$$

(essentially), where not all of the c's are zero. Multiplying this relation in succession by $\bar{\varphi}_1, \bar{\varphi}_2, \cdots, \bar{\varphi}_n$, and integrating from a to b we get the n equations

$$c_1\int \varphi_1\bar{\varphi}_i + c_2\int \varphi_2\bar{\varphi}_i + \cdots + c_n\int \varphi_n\bar{\varphi}_i = 0 \qquad (i = 1, 2, \cdots, n).$$

Since the c's are not all zero the determinant of this system must vanish; thus, interchanging rows and columns in this determinant, we have

$$(2) \qquad \begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \int \varphi_1\bar{\varphi}_2 & \cdots & \int \varphi_1\bar{\varphi}_n \\ \int \varphi_2\bar{\varphi}_1 & \int \varphi_2\bar{\varphi}_2 & \cdots & \int \varphi_2\bar{\varphi}_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \int \varphi_n\bar{\varphi}_2 & \cdots & \int \varphi_n\bar{\varphi}_n \end{vmatrix} = 0.$$

This relation is therefore a necessary condition for essential linear dependence.

It is also a sufficient condition. For suppose that (2) is fulfilled; then the n sets of constants forming the rows of the determinant are linearly dependent, and we have

$$\int \bar{\varphi}_i(c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n) = 0 \quad (i = 1, 2, \cdots, n),$$

where not all of the c's vanish. From Theorem 1 we now readily infer that

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n = 0$$

(essentially), which establishes the essential linear dependence of $\varphi_1, \varphi_2, \cdots, \varphi_n$.

We term the determinant in (2) the *Gramian* of $\varphi_1, \varphi_2, \cdots, \varphi_n$ and denote it by $G(\varphi_1, \varphi_2, \cdots, \varphi_n)$. Thus we have

THEOREM 2. *A necessary and sufficient condition that the n functions*

$\varphi_1, \varphi_2, \cdots, \varphi_n$ *of the class* Ω *be essentially linearly dependent is that their Gramian vanish.**

We turn now to the solution of the system (1), assuming that $\varphi_1, \varphi_2, \cdots, \varphi_n$ are essentially linearly independent, so that $G(\varphi_1, \cdots, \varphi_n) \neq 0$. This entails no loss in generality; since if, for example, φ_n were essentially linearly dependent on $\varphi_1, \cdots, \varphi_{n-1}$, the equation $\int \varphi_n \xi = 0$ would be satisfied by all the solutions of the preceding system and could therefore be omitted. Every solution ξ_1 of (1) of the class Ω may be written in the form

$$\xi_1 = c_1\bar{\varphi}_1 + \cdots + c_n\bar{\varphi}_n + \bar{\eta}, \tag{3}$$

η being some function of the class Ω. In order that this be a solution the constants c_i must satisfy the system of linear equations

$$c_1\int \varphi_i\bar{\varphi}_1 + \cdots + c_n\int \varphi_i\bar{\varphi}_n + \int \varphi_i\bar{\eta} = 0 \quad (i = 1, 2, \cdots, n),$$

which may be solved by Cramer's rule as their determinant is precisely $G(\varphi_1, \cdots, \varphi_n)$. These values substituted in (3) give

$$\xi_1 = \frac{\begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_n & \int \varphi_1\bar{\eta} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_n & \int \varphi_n\bar{\eta} \\ \bar{\varphi}_1 & \cdots & \bar{\varphi}_n & \bar{\eta} \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}. \tag{4}$$

That, no matter what the function η of the class Ω may be, the formula (4) always gives a solution of (1) belonging to this class is seen by direct substitution.

THEOREM 3. *If in equations* (1) $\varphi_1, \varphi_2, \cdots, \varphi_n$ *are essentially linearly independent functions belonging to the class* Ω, *their general solution of this class is given by* (4), η *being an arbitrary function of the class* Ω.†

A line of reasoning analogous to that on p. 171 of (I) shows that two different $\bar{\eta}$'s yield essentially the same ξ_1 when and only when their difference is essentially linearly dependent upon $\bar{\varphi}_1, \bar{\varphi}_2, \cdots, \bar{\varphi}_n$.

To find a formula for norm ξ_1 we form $\int \xi_1\bar{\xi}_1$ from (3), whence, remembering that the norm is real,

$$\text{norm } \xi_1 = \int \eta\xi_1; \tag{5}$$

* We note in passing that $\varphi_1, \varphi_2, \cdots, \varphi_n$ are connected by the same linear relation that connects the rows of their Gramian, *written as above*.

† If η is given the solution ξ_1 is essentially unique.

and from (4) the latter integral is seen to be equal to

$$\text{norm } \xi_1 = \frac{G(\varphi_1, \cdots, \varphi_n, \eta)}{G(\varphi_1, \cdots, \varphi_n)}. \tag{6}$$

Just as in (I), p. 171, this relation may be used to establish

THEOREM 4. *The Gramian of any number of essentially linearly independent functions of the class Ω is real and positive.*

3. We now pass to the non-homogeneous system

$$\int \varphi_1 \xi = b_1, \qquad \int \varphi_2 \xi = b_2, \qquad \cdots, \qquad \int \varphi_n \xi = b_n, \tag{7}$$

where we again assume that $\varphi_1, \cdots, \varphi_n$ are essentially linearly independent functions of the class Ω. We seek a solution of the form

$$\xi_0 = c_1 \bar{\varphi}_1 + \cdots + c_n \bar{\varphi}_n. \tag{8}$$

Substituting this in (7) we obtain n linear equations to determine the c's. These can, as above, be solved by Cramer's rule; and the result substituted in (8) gives

$$\xi_0 = \frac{\begin{vmatrix} \int \varphi_1 \bar{\varphi}_1 & \cdots & \int \varphi_1 \bar{\varphi}_n & -b_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n \bar{\varphi}_1 & \cdots & \int \varphi_n \bar{\varphi}_n & -b_n \\ \bar{\varphi}_1 & \cdots & \bar{\varphi}_n & 0 \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}. \tag{9}$$

That this is really a solution of (7) we see by direct substitution. Hence we may state

THEOREM 5. *If $\varphi_1, \varphi_2, \cdots, \varphi_n$ are essentially linearly independent functions of the class Ω, the equations* (7) *have one and only one solution of the form* (8), *and this is given by* (9).

The general solution of (7) is obtained by adding to the particular solution (9) the general solution (4) of the homogeneous system (1); it is therefore

$$\xi = \xi_0 + \xi_1 = \frac{\begin{vmatrix} \int \varphi_1 \bar{\varphi}_1 & \cdots & \int \varphi_1 \bar{\varphi}_n & \int \varphi_1 \bar{\eta} - b_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n \bar{\varphi}_1 & \cdots & \int \varphi_n \bar{\varphi}_n & \int \varphi_n \eta - b_n \\ \bar{\varphi}_1 & \cdots & \bar{\varphi}_n & \bar{\eta} \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}. \tag{10}$$

The solution (9) of (7), which is characterized by being the only solution of (7) which is linearly dependent upon the $\bar{\varphi}$'s, shall be called the *principal*

solution of (7). Another characteristic property of ξ_0, which can be readily deduced from (10) and (8), is that

$$\text{norm } \xi = \text{norm } \xi_0 + \text{norm } \xi_1 \tag{11}$$

and hence

$$\text{norm } \xi \geqq \text{norm } \xi_0,$$

the equality sign holding only when ξ_1 is essentially zero, in which case ξ and ξ_0 are essentially equal. Thus we have

THEOREM 6. *Among the solutions of* (7) *no others have so small a norm as the principal solution except those essentially equal to it.*

To obtain a formula for norm ξ_0 we form $\int \xi_0\bar{\xi}_0$ from (9):*

$$\text{norm } \xi_0 = -\frac{\begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_n & b_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_n & b_n \\ \bar{b}_1 & \cdots & \bar{b}_n & 0 \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}. \tag{12}$$

Norm ξ is now given by (11).

Let us now write those particular c_i's which when substituted in (8) give ξ_0 as c_i^0, so that

$$\xi_0 = c_1^0\bar{\varphi}_1 + \cdots + c_n^0\bar{\varphi}_n.$$

Then

$$\text{norm } \xi_0 = \int \xi_0\bar{\xi}_0 = \sum_{i=1}^{n} c_i^0\bar{b}_i. \tag{13}$$

We now inquire as to what special role the solution ξ_0 plays in the totality of expressions essentially of the form $\Sigma_{i=1}^n c_i\bar{\varphi}_i$, regarded as functions of the c's, which may vary subject to the condition

$$\text{norm} \sum_{i=1}^{n} c_i\bar{\varphi}_i = \sum_{i=1}^{n} c_i\bar{b}_i \tag{14}$$

suggested by (13). For a convenience in the following argument we will exclude the case $c_1 = c_2 = \cdots = c_n = 0$; then, as the φ's are essentially linearly independent, (14) requires, in particular, that $\Sigma_{i=1}^n c_i\bar{b}_i$ be always real and positive. Letting ξ represent an arbitrary solution of (7), we have from Theorems 2 and 4

* As in (I), p. 173, footnote, we may use (12) to prove the THEOREM: *If the Gramian of essentially linearly independent functions of the class Ω is bordered by constants, which do not all vanish, so as to form a bordered Gramian of the type of that in* (12), *this bordered Gramian is negative.*

$$G(\xi, \Sigma c_i\bar{\varphi}_i) = \begin{vmatrix} \int |\xi|^2 & \int \xi\Sigma\bar{c}_i\varphi_i \\ \int \bar{\xi}\Sigma c_i\bar{\varphi}_i & \int |\Sigma c_i\bar{\varphi}_i|^2 \end{vmatrix} \geqq 0,$$

or

$$\int |\xi|^2 \cdot \int |\Sigma c_1\bar{\varphi}_i|^2 \geqq [\Sigma c_i\bar{b}_i]^2.$$

In view of (14) this yields

(15) $$\text{norm } \xi \geqq \Sigma c_i\bar{b}_i,$$

the sign of equality holding when and only when ξ and $\Sigma c_i\bar{\varphi}_i$ are essentially linearly dependent. This is clearly not the case when ξ is essentially *not* of the form $\Sigma c_i\bar{\varphi}_i$, i. e., when ξ is essentially different from ξ_0; hence writing $c_i = c_i^0$ in (15), we have that norm $\xi >$ norm ξ_0 except when ξ and ξ_0 are essentially equal. This constitutes another proof of Theorem 6.

Again, letting $\xi = \xi_0$ in (15), we have

$$\text{norm } \xi_0 \geqq \Sigma c\bar{b}_i,$$

the equality sign holding only when

$$\xi_0 = k\Sigma c_i\bar{\varphi}_i$$

(essentially), in which case we find from (14)

$$\text{norm } \xi_0 = |k|^2\Sigma c_i\bar{b}_i = k\Sigma c_i\bar{b}_i.$$

Hence $k = 1$, the value $k = 0$ being excluded as ξ_0 is not essentially zero, and

$$\Sigma(c_i^0 - c_i)\bar{\varphi}_i = 0$$

(essentially), which, in view of essential linear independence of the $\bar{\varphi}$'s, can only be true when $c_i^0 = c_i$. Thus we have shown that the polynomial $\Sigma c_i\bar{b}_i$ in $c_1, \cdots, c_n$, when subject to the condition (14) attains its maximum value, norm ξ_0, when and only when $c_i = c_i^0$.

THEOREM 7. *In the principal solution*

$$\xi_0 = c_1^0\bar{\varphi}_1 + \cdots + c_n^0\bar{\varphi}_n$$

the constants $(c_1^0, \cdots, c_n^0)$ *form that set of values which render the polynomial* $c_1\bar{b}_1 + \cdots + c_n\bar{b}_n$ *a maximum when it is subject to the condition* (14).

Thus the finding of the principal solution of equations (7) may be regarded as a problem in conditional maxima, and in fact ξ_0 may be obtained by the classical procedure for such problems.*

4. Before giving an application of the preceding results we lay down the following definitions:

* See F. Riesz, l. c., § 9, where a more general problem is treated in this way.

A function of the class Ω is said to be *normalized* when its norm is unity.

Two functions φ, ψ are said to be *orthogonal* in the interval $a \leqq x \leqq b$ if $\int_a^b \varphi\bar{\psi}dx$, and hence also $\int_a^b \bar{\varphi}\psi dx$, is zero. The system of functions $\{\varphi_i\}$, finite or infinite in number, is termed orthogonal in the interval $a \leqq x \leqq b$ when

$$\int_a^b \varphi_i\bar{\varphi}_j dx = \int_a^b \bar{\varphi}_i\varphi_j dx = 0 \qquad (i \neq j).$$

We now propose to solve the problem: *Given the system of functions $\{\varphi_n\}$ of the class Ω, none of which are essentially linearly dependent, to find a system of normalized, orthogonal functions $\{\Phi_n\}$ such that*

$$\Phi_n = \sum_{i=1}^{i=n} a_{ni}\varphi_i, \qquad a_{nn} \neq 0,$$

the a's being constants.

Φ_n must satisfy the n equations

$$\int \bar{\Phi}_i\xi = \begin{cases} 0 & (i \neq n), \\ 1 & (i = n); \end{cases}$$

and upon substituting for $\bar{\Phi}_i$ the linear expression in the $\bar{\varphi}$'s an evident reduction yields the equivalent system

$$\int \bar{\varphi}_1\xi = 0, \qquad \int \bar{\varphi}_2\xi = 0, \qquad \cdots, \qquad \int \bar{\varphi}_{n-1}\xi = 0, \qquad \int \bar{\varphi}_n\xi = \frac{1}{\bar{a}_{nn}}.$$

From Theorem 5 we see that the only solution of this system having the form of Φ_n is its principal solution ξ_0. To determine a_{nn} we have from (12)

$$\text{norm } \xi_0 = \frac{1}{|a_{nn}|^2} \cdot \frac{G_{n-1}}{G_n} = 1,$$

$$|a_{nn}| = \sqrt{\frac{G_{n-1}}{G_n}}, \tag{16}$$

where we have written $G_n = G(\varphi_1, \cdots, \varphi_n) = G(\bar{\varphi}_1, \cdots, \bar{\varphi}_n)$ with the convention that $G_0 = 1$. Thus, choosing a_{nn} in accordance with (16), we may write $\Phi_n = \xi_0$. However Φ_n is not uniquely determined for there are infinitely many complex numbers satisfying (16); but if we impose the further condition that a_{nn} be real and positive $a_{nn} = \sqrt{G_{n-1}/G_n}$ and Φ_n will be uniquely determined. Then we have from (9), after interchanging rows and columns in the determinant in the numerator,*

* These formulæ are given by Kowalewski, Einführung in die Determinantentheorie, Leipzig (1909), p. 337.

$$(17)\qquad \Phi_n = \frac{1}{\sqrt{G_{n-1}G_n}}\begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_{n-1} & \varphi_1 \\ \int \varphi_2\bar{\varphi}_1 & \cdots & \int \varphi_2\bar{\varphi}_{n-1} & \varphi_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_{n-1} & \varphi_n \end{vmatrix} \qquad (n = 1, 2, \cdots).$$

Denoting the cofactor of the element in the last column and ith row of G_n by $G_n^{(i)}$, it is clear that

$$(18)\qquad a_{ni} = \frac{G_n^{(i)}}{\sqrt{G_{n-1}G_n}} \qquad (i = 1, 2, \cdots, n).$$

Thus the problem is solved; if we require that a_{nn} be real and positive there is only one solution and this is given by (17).

5.* We proceed to establish two important inequalities. If φ and ψ are functions of the class Ω we have from Theorems 2 and 4 that $G(\varphi, \bar{\psi}) \geqq 0$, which upon expansion gives Schwarz's Inequality:

$$(19)\qquad \left|\int \varphi\psi\right| \leqq \left[\int |\varphi|^2\right]^{\frac{1}{2}}\left[\int |\psi|^2\right]^{\frac{1}{2}}.$$

Applying this result to $\int (\varphi + \psi)(\bar{\varphi} + \bar{\psi})$ we deduce further that

$$(20)\qquad \left[\int |\varphi + \psi|^2\right]^{\frac{1}{2}} \leqq \left[\int |\varphi|^2\right]^{\frac{1}{2}} + \left[\int |\psi|^2\right]^{\frac{1}{2}}.$$

We next lay down the following

DEFINITION. *The sequence of functions $\{\varphi_n\}$ of the class Ω is said to converge in the mean to the function φ of this class if*

$$\lim_{n=\infty} \int |\varphi - \varphi_n|^2 = 0,\dagger$$

and we write $\underline{\lim}_{n=\infty}\, \varphi_n = \varphi$.

The function φ is essentially unique; that is, if $\{\varphi_n\}$ converges in the mean to both φ and ψ, these functions are essentially equal. For by (20)

$$\left[\int |\varphi - \psi|^2\right]^{\frac{1}{2}} = \left[\int |\varphi - \varphi_n + \varphi_n - \psi|^2\right]^{\frac{1}{2}} \leqq \left[\int |\varphi - \varphi_n|^2\right]^{\frac{1}{2}} + \left[\int |\varphi_n - \psi|^2\right]^{\frac{1}{2}},$$

and upon passing to the limit $n = \infty$ we have $\int |\varphi - \psi|^2 = 0$.

It is clear that uniform convergence implies mean convergence. On

* This article is closely analogous to § 5 of (I), to which reference should be made for details of proofs.

† E. Fischer, "Sur la convergence en moyenne," Comptes Rendus, May, 1907, p. 1022. For a generalization of this conception see F. Riesz, l. c., p. 464.

We note also that if the sequence $\{\varphi_n\}$ of the class Ω fulfils this relation, φ is necessarily of the class Ω. For the definition implies that $\varphi - \varphi_n$ is of the class Ω for sufficiently large values of n, and hence the sum of $\varphi - \varphi_n$ and φ_n is also of this class.

the contrary, mean convergence does not even imply convergence in the ordinary sense.

If $\underline{\lim}_{n=\infty} \varphi_n = \varphi$, $\underline{\lim}_{n=\infty} \psi_n = \psi$, it follows from (20) that

$$\underline{\lim}_{n=\infty} (\varphi_n + \psi_n) = \varphi + \psi. \tag{21}$$

Furthermore

$$\lim_{n=\infty} \int \varphi_n \psi_n = \varphi\psi \tag{22}$$

as we readily prove upon writing

$$\int (\varphi\psi - \varphi_n\psi_n) = \int (\varphi - \varphi_n)\psi + \int (\psi - \psi_n)\varphi - \int (\varphi - \varphi_n)(\psi - \psi_n)$$

and applying (19). Important special cases of (22) are

$$\lim_{n=\infty} \int \varphi_n\psi = \int \varphi\psi; \tag{23}$$

$$\lim_{n=\infty} \text{norm } \varphi_n = \text{norm } \varphi. \tag{24}$$

We now state without proof an important criterion for mean convergence due to E. Fischer.*

THEOREM 8. *A necessary and sufficient condition that the sequence of functions* $\{\varphi_n\}$ *of the class* Ω *converge in the mean to a function of this class is that to every positive* ϵ *there correspond an integer* N *such that*

$$\int |\varphi_n - \varphi_m|^2 < \epsilon, \qquad m, n > N.\dagger$$

The infinite series of functions of the class Ω, $\varphi_1 + \varphi_2 + \cdots$, is said to converge in the mean to the function σ when the sum of its first n terms, σ_n, converges in the mean to σ. From Theorem 8 we see that a necessary and sufficient condition for the mean convergence of the above series is that, to every positive ϵ, there correspond an integer N such that

$$\int |\sigma_n - \sigma_m|^2 = \int |\varphi_{m+1} + \varphi_{m+2} + \cdots + \varphi_n|^2 < \epsilon, \qquad m, n > N. \tag{25}$$

If the φ's are all mutually orthogonal (25) becomes

$$\int |\varphi_{m+1}|^2 + \int |\varphi_{m+2}|^2 + \cdots + \int |\varphi_n|^2 < \epsilon, \qquad m, n > N;$$

and as this is precisely a necessary and sufficient condition that the series $\int |\varphi_1|^2 + \int |\varphi_2|^2 + \cdots$ converge, we have proved

* Sur la convergence en moyenne, l. c.

† The apparently redundant phrase "to a function of the class Ω" is inserted because E. Fischer *defines* the sequence to be convergent in the mean when this relation holds and then establishes the existence of a function of the class Ω to which the sequence converges in the mean.

For other proofs of this theorem see Weyl, "Über die Konvergenz von Reihen die nach Orthogonalfunktionen fortschreiten," Mathematische Annalen, 67 (1909), p. 243; Plancherel, "Contribution à l'étude de la représentation d'une fonction arbitraire," etc., Rend. Circ. Matem. Palermo, 30 (1910), p. 292. See also F. Riesz, l. c., p. 468, where a generalization of this theorem is proved.

THEOREM 9. *A series of mutually orthogonal functions of the class Ω converges in the mean to a function σ of this class when and only when the series of their norms converges.**

Moreover as

$$\int |\sigma_n|^2 = \int |\varphi_1|^2 + \int |\varphi_2|^2 + \cdots + \int |\varphi_n|^2$$

we infer from (24) the

COROLLARY. *If the conditions of Theorem 9 are fulfilled, the norm of the function σ is equal to the series of the norms of the terms.*

6. We are now in position to consider the infinite system of homogeneous equations

$$\int \varphi_1\xi = 0, \qquad \int \varphi_2\xi = 0, \qquad \cdots, \tag{26}$$

where we assume that $\varphi_1, \varphi_2, \cdots$ are all of the class Ω and none of them are essentially linearly dependent. The general solution of the first n of these equations, which we denote by $\xi_1^{(n)}$, is given by formula (4). We now express $\xi_1^{(m)}$ as the sum of m terms of a series, writing

$$\xi_1^{(m)} = \xi_1^{(1)} + \sum_{n=2}^{n=m} (\xi_1^{(n)} - \xi_1^{(n-1)});$$

and by means of a device entirely analogous to that used in (I), pp. 179, 180, we find that†

$$\xi_1^{(m)} = \bar{\eta} - \sum_{n=1}^{n=m} \frac{H_n}{G_{n-1}G_n}\psi_n, \tag{27}$$

where we have written

$$\psi_1 = \bar{\varphi}_1, \qquad \psi_n = \begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_n \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \int \varphi_{n-1}\bar{\varphi}_1 & \cdots & \int \varphi_{n-1}\bar{\varphi}_n \\ \bar{\varphi}_1 & \cdots & \bar{\varphi}_n \end{vmatrix} \qquad (n = 2, 3, \cdots),$$

$$H_1 = \int \varphi_1\bar{\eta}, \qquad H_n = \begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_{n-1} & \int \varphi_1\bar{\eta} \\ \int \varphi_2\bar{\varphi}_1 & \cdots & \int \varphi_2\bar{\varphi}_{n-1} & \int \varphi_2\bar{\eta} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_{n-1} & \int \varphi_n\bar{\eta} \end{vmatrix} \qquad (n = 2, 3, \cdots),$$

* It is easily seen that this theorem still holds if the condition of orthogonality is replaced by the less restrictive requirement that the real part of $\int \varphi_i\bar{\varphi}_j$ vanish, i. e.,

$$\int \varphi_i\bar{\varphi}_j + \int \bar{\varphi}_i\varphi_j = 0 \qquad (i \neq j).$$

† Cf. equation (30) of (I). The notation used above corresponds with that of (I).

$$G_0 = 1, \qquad G_n = G(\varphi_1, \cdots, \varphi_n), \quad (n = 1, 2, 3, \cdots).$$

The ψ's are all mutually orthogonal functions of the class Ω; for as

$$\int \psi_n \varphi_i = \int \bar{\psi}_n \bar{\varphi}_i = 0 \qquad (i = 1, 2, \cdots, n-1),$$

we have

$$\int \psi_n \bar{\psi}_m = 0 \qquad (m \neq n).$$

Moreover we have the relations

$$\int |\psi_n|^2 = \int \psi_n \bar{\psi}_n = \int \psi_n \varphi_n G_{n-1} = G_n G_{n-1},$$

$$\int \psi_n \eta = \bar{H}_n, \qquad \int \bar{\psi}_n \bar{\eta} = H_n,$$

so that (27) may be put in the form

$$\xi_1^{(m)} = \bar{\eta} - \sum_{n=1}^{n=m} \frac{\int \bar{\psi}_n \bar{\eta}}{\int |\psi_n|^2} \psi_n. \tag{28}$$

From (5) we have

$$\text{norm } \xi_1^{(m)} = \int \eta \xi_1^{(m)} = \int |\eta|^2 - \sum_{n=1}^{n=m} \frac{\left|\int \psi_n \eta\right|^2}{\int |\psi_n|^2},$$

which shows that the series of positive or zero terms

$$\sum_{n=1}^{\infty} \frac{\left|\int \psi_n \eta\right|^2}{\int |\psi_n|^2}$$

is convergent for every function η of the class Ω since the sum of its first m terms is not greater than $\int |\eta|^2$. Now this series is precisely that formed by the norms of the terms of the series

$$\sum_{n=1}^{\infty} \frac{\int \bar{\psi}_n \bar{\eta}}{\int |\psi_n|^2} \psi_n, \tag{29}$$

whose terms are all mutually orthogonal; hence by Theorem 9 the series (29) converges in the mean to a function of the class Ω when η is a function of this class, as does likewise the series obtained from (28) by extending the summation to $n = \infty$. Thus we have shown the existence of a function $\xi_1 = \lim_{m=\infty} \xi_1^{(m)}$, and from (23) it is clear that ξ_1 is a solution of equations (26). Conversely, if ξ_1 is any solution of these equations we may obtain it by putting $\bar{\eta} = \xi_1$ in (28); for then all the terms after the first vanish. We therefore have

THEOREM 10. *If η is a function of the class Ω, the function $\xi_1^{(n)}$ given by formula* (4) *converges in the mean as n becomes infinite to a function ξ_1*

of the class Ω *which is a solution of equations* (26). *Conversely, every solution of* (26), *whether of the class* Ω *or not, can be obtained in this way by properly choosing* η.

The corollary to Theorem 9 shows that

$$\text{(30)} \qquad \operatorname{norm} \xi_1 = \operatorname{norm} \eta - \sum_{n=1}^{\infty} \frac{\left|\int \psi_n \eta\right|^2}{\int |\psi_n|^2}.$$

Referring to (6) we see that we may also write

$$\text{(30)}' \qquad \operatorname{norm} \xi_1 = \lim_{n=\infty} \frac{G(\eta, \varphi_1, \cdots, \varphi_n)}{G(\varphi_1, \cdots, \varphi_n)}.$$

DEFINITION. *A system of functions of the class* Ω *is termed complete in the interval* (a, b) *if there exists no other function of this class, not essentially zero, which is orthogonal to all the functions of the system in this interval.*

If the system $\{\varphi_n\}$ is complete the equations (26) have no solution not essentially zero; then norm $\xi_1 = 0$, and in view of (30) we have

THEOREM 11. *A necessary and sufficient condition that the system of functions* $\{\varphi_n\}$ *of the class* Ω *be complete is that, for any function* η *of this class, we have*

$$\text{(31)} \qquad \operatorname{norm} \eta = \sum_{n=1}^{\infty} \frac{\left|\int \psi_n \eta\right|^2}{\int |\psi_n|^2},$$

or

$$\text{(32)} \qquad \lim_{n=\infty} \frac{G(\eta, \varphi_1, \cdots, \varphi_n)}{G(\varphi_1, \cdots, \varphi_n)} = 0.$$

In terms of the normalized orthogonal system $\{\Phi_n\}$ formed from $\{\varphi_n\}$ as indicated by (17), the condition (31) takes the form

$$\text{(33)} \qquad \operatorname{norm} \eta = \sum_{n=1}^{\infty} \left|\int \Phi_n \bar{\eta}\right|^2$$

since

$$\frac{\left|\int \psi_n \eta\right|^2}{\int |\psi_n|^2} = \frac{|H_n|^2}{G_{n-1} G_n} = \left|\int \Phi_n \bar{\eta}\right|^2.$$

If the system $\{\varphi_n\}$ is incomplete we see from (30) that

$$\text{(34)} \qquad \operatorname{norm} \eta \geqq \sum_{n=1}^{\infty} \left|\int \Phi_n \bar{\eta}\right|^2$$

so that the series on the right is convergent whenever η is of the class Ω.

In the important case in which $\{\varphi_n\}$ is a normalized orthogonal system we may put $\Phi_n = \varphi_n$ in (33) and (34), which then yield the well-known *Vollständigkeitsbedingung* and *Bessel's Inequality* respectively.

7. We proceed to apply the foregoing results to the problem of expanding an arbitrary function in terms of a preassigned system of functions by the method of least squares.*

THEOREM 12.† *Let θ be any function of the class Ω and $\{\varphi_k\}$ a complete system of functions of this class none of which are essentially linearly dependent. Then if the constants a_{nk} in the finite series*

$$\sigma_n = \sum_{k=1}^{k=n} a_{nk}\varphi_k \qquad (n = 1, 2, \cdots),$$

are chosen so that norm $(\theta - \sigma_n)$ has the smallest value,

$$\lim_{n=\infty} \sigma_n = \theta.$$

Writing

$$a_{nk} = a_{nk}' + ia_{nk}'', \qquad \bar{a}_{nk} = a_{nk}' - ia_{nk}'' \quad (k = 1, 2, \cdots, n),$$

and employing the classical procedure to find the values of the $2n$ constants, a_{nk}', a_{nk}'' which render

$$I_n = \text{norm } (\theta - \sigma_n) = \int (\theta - \sigma_n)(\bar{\theta} - \bar{\sigma}_n)$$

a minimum, we find that

$$(35) \qquad \begin{cases} \dfrac{\partial I_n}{\partial a_{nk}'} = -\int [(\theta - \sigma_n)\bar{\varphi}_k + (\bar{\theta} - \bar{\sigma}_n)\varphi_k] = 0, \\ \dfrac{\partial I_n}{\partial a_{nk}''} = i\int [(\theta - \sigma_n)\bar{\varphi}_k - (\bar{\theta} - \bar{\sigma}_n)\varphi_k] = 0; \end{cases} \qquad (k = 1, 2, \cdots, n)$$

$$(36) \qquad \frac{\partial^2 I_n}{\partial a_{nk}'^2} = \frac{\partial^2 I_n}{\partial a_{nk}''^2} = 2\int \varphi_k\bar{\varphi}_k > 0, \qquad \frac{\partial^2 I_n}{\partial a_{nk}'\partial a_{nk}''} = 0,$$

noticing that φ_k is not essentially zero as none of the φ's are essentially linearly dependent. Now equations (35) are equivalent to

$$\begin{aligned} \int (\theta - \sigma_n)\bar{\varphi}_k &= 0, \\ \int (\bar{\theta} - \bar{\sigma}_n)\varphi_k &= 0, \end{aligned} \qquad (k = 1, 2, \cdots, n)$$

either set of which follows from the other. Thus $\bar{\theta} - \sigma_n$, which is a linear combination of $\bar{\theta}$ and the $\bar{\varphi}$'s, is a solution of the homogeneous equations $\int \varphi_k\xi = 0$ $(k = 1, 2, \cdots, n)$, and is therefore uniquely determined by formula (4) upon putting $\eta = \theta$. In view of (36) this expression for $\bar{\theta} - \bar{\sigma}_n$ gives I_n its minimum value, which by (6) is

* See Gram, Über die Entwickelung reeler Funktionen in Reihen mittelst der Methode der kleinsten Quadrate, Crelle, 94 (1883), pp. 41–73.

Byerly, "Approximate Representation," Annals of Mathematics, Ser. 2, 12 (1911), pp. 128–148.

† Cf. Gram, l. c., p. 59.

$$\text{norm}\,(\theta - \sigma_n) = \frac{G(\theta, \varphi_1, \cdots, \varphi_n)}{G(\varphi_1, \cdots, \varphi_n)}.$$

Since the system $\{\varphi_k\}$ is complete we have from Theorem 11 that

$$\lim_{n=\infty} \text{norm}\,(\theta - \sigma_n) = 0,$$

or, in our notation, $\underset{n=\infty}{\underline{\underline{\lim}}}\, \sigma_n = \theta$, as we wished to prove.

8. Lastly we consider the infinite system of non-homogeneous equations

$$\int \varphi_1 \xi = b_1, \qquad \int \varphi_2 \xi = b_2, \qquad \cdots, \tag{37}$$

where we again assume that $\varphi_1, \varphi_2, \cdots$ are all of the class Ω and none of them essentially linearly dependent. The principal solution of the first n of these equations, which we will denote by $\xi_0^{(n)}$, is given by formula (9), and an argument similar to that used in (I), p. 182, shows that we may write $\xi_0^{(m)}$ as the sum of m mutually orthogonal functions of the class Ω,

$$\xi_0^{(m)} = \sum_{n=1}^{n=m} \frac{B_n}{G_{n-1}G_n} \psi_n,$$

where ψ_n is defined as in § 6 and

$$B_1 = b_1, \qquad B_n = \begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_{n-1} & b_1 \\ \int \varphi_2\bar{\varphi}_1 & \cdots & \int \varphi_2\bar{\varphi}_{n-1} & b_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_{n-1} & b_n \end{vmatrix} \qquad (n = 2, 3, \cdots).$$

Theorem 9 shows that $\xi_0^{(m)}$ will converge in the mean to a function of the class Ω when and only when the norms of the terms of the series

$$\sum_{n=1}^{\infty} \frac{B_n}{G_{n-1}G_n} \psi_n$$

form a convergent series. Thus when

$$\sum_{n=1}^{\infty} \frac{|B_n|^2}{G_{n-1}G_n} \tag{38}$$

converges there exists a function $\xi_0 = \underset{n=\infty}{\underline{\underline{\lim}}}\, \xi_0^{(n)}$ of the class Ω, and from (23) we see that ξ_0 is a solution of (37). Now if the equations (37) have *any* solution, ξ, of the class Ω, then, $\xi_0^{(m)}$ being the solution of least norm of the first m of these equations,

$$\text{norm}\, \xi_0^{(m)} \leqq \text{norm}\, \xi;$$

and as norm $\xi_0^{(m)}$ is precisely the sum of the first m terms of series (38), whose terms are never negative, this series converges. Thus we have proved

THEOREM 13. *A necessary and sufficient condition that equations* (37) *have a solution of the class Ω is that series* (38) *converge. When this is the case $\xi_0^{(n)}$, given by formula* (9), *converges in the mean to a function of the class Ω as n becomes infinite, and this function ξ_0 is a solution of equations* (37).

ξ_0 is termed the *principal solution* of (37). We may form the general solution ξ by adding to ξ_0 the general solution ξ_1 of equations (26): $\xi = \xi_0 + \xi_1$. If the system $\{\varphi_n\}$ is complete ξ_1 is essentially zero; then if the series (38) converge the equations (37) have essentially but one solution, the principal solution.

If ξ_1 is a function of the class Ω, ξ is likewise and its norm is given by

$$\text{norm } \xi = \text{norm } \xi_0 + \text{norm } \xi_1;$$

for from (8) we have $\int \xi_0^{(n)} \bar{\xi}_1 = 0$, so that upon applying (23), $\int \xi_0 \bar{\xi}_1 = 0$. Thus we have

THEOREM 14. *Among the solutions of* (37) *no others have so small a norm as the principal solution except those essentially equal to it.*

The norm of ξ_0 is given by the series (38), as is clear from the Corollary to Theorem 9; or we may write

$$\text{(39)} \qquad \text{norm } \xi_0 = \lim_{n=\infty} - \frac{\begin{vmatrix} \int \varphi_1 \bar{\varphi}_1 & \cdots & \int \varphi_1 \bar{\varphi}_n & b_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n \bar{\varphi}_1 & \cdots & \int \varphi_n \bar{\varphi}_n & b_n \\ \bar{b}_1 & \cdots & \bar{b}_n & 0 \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}.$$

An important case arises when, in (37), the functions $\{\varphi_n\}$ form a normalized orthogonal system, when, in particular, none of them will be essentially linearly dependent. Then $B_n = b_n$ and series (38) becomes $\Sigma |b_n|^2$; Theorem 13 for this case is the *Theorem of Riesz and Fischer.**

DEFINITION. *Two systems of functions, $\{\varphi_n\}$ and $\{\psi_n\}$, of the class Ω are said to form a biorthogonal system $\{\varphi_n, \psi_n\}$ in the interval $a \leqq x \leqq b$ if*

$$\int_a^b \varphi_i \bar{\psi}_j dx = \int_a^b \bar{\varphi}_i \psi_j dx = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j). \end{cases}$$

Each of the systems $\{\varphi_n\}$, $\{\psi_n\}$ is called the adjoint of the other.

* F. Riesz, Comptes Rendus, April, 1907, p. 734; E. Fischer, Comptes Rendus, May, 1907, p. 1022.

If $\{\varphi_n, \psi_n\}$ is a biorthogonal system of functions, none of the functions of $\{\varphi_n\}$ or of $\{\psi_n\}$ can be essentially linearly dependent; if, for example,

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_k\varphi_k = 0$$

(essentially), we have, upon multiplying this relation by $\bar{\psi}_i$ and integrating from a to b, $c_i = 0$ $(i = 1, 2, \cdots, k)$.

Now let $\{\varphi_n\}$ be a system of functions of the class Ω none of which are essentially linearly dependent. We inquire under what condition $\{\varphi_n\}$ will have an adjoint system; or when the system of equations

$$\int \varphi_i \xi = \begin{cases} 1 & (i = n), \\ 0 & (i \neq n), \end{cases} \tag{40}$$

has a solution of the class Ω for $n = 1, 2, 3, \cdots$. A necessary and sufficient condition that (40) have such a solution is that the series (38) converge for

$$b_i = \begin{cases} 1 & (i = n), \\ 0 & (i \neq n); \end{cases}$$

this series, in the notation of § 4, reduces to

$$\frac{G_{n-1}}{G_n} + \frac{|G_{n+1}{}^{(n)}|^2}{G_n G_{n+1}} + \frac{|G_{n+2}{}^{(n)}|^2}{G_{n+1} G_{n+2}} + \cdots$$

or with regard to (18),

$$a_{nn}{}^2 + |a_{n+1,\,n}|^2 + |a_{n+2,\,n}|^2 + \cdots. \tag{41}$$

Thus we have proved a theorem due to A. J. Pell:* *A necessary and sufficient condition that a system of function $\{\varphi_n\}$ of the class Ω, none of which are essentially linearly dependent, have an adjoint system is that the series* (41) *converge for $n = 1, 2, \cdots$.*

When the conditions of this theorem are fulfilled and the system $\{\varphi_n\}$ is complete the adjoint system will be essentially unique.

9. The expressions (30)′ and (39) for the norms of ξ_1 and ξ_0 suggest the problem of determining under what conditions the determinants in the numerator and denominator converge as n becomes infinite. As the quotient of these determinants converges we need merely inquire when the *infinite Gramian* $G(\varphi_1, \varphi_2, \cdots)$ converges. By means of Theorem 4 and the theorem concerning bordered Gramians stated in the footnote to equation (12) we may prove just as in (I), § 7, the following

* "Biorthogonal Systems of Functions," Transactions of the American Mathematical Society, 12 (1911), p. 141.

THEOREM 15. *A sufficient condition that the infinite Gramian of a system of essentially linearly independent functions* $\{\varphi_n\}$ *of the class* Ω *converge is that the infinite product* $\Pi_{n=1}^{\infty} \int |\varphi_n|^2$ *diverge to zero or converge.*

From the proof of this theorem the following corollaries follow immediately.

COROLLARY 1. *If the infinite Gramian of the normalized system of functions formed from* $\{\varphi_n\}$ *does not diverge to zero the condition that* $\Pi_{n=1}^{\infty} |\varphi_n|^2$ *diverge to zero or converge is also necessary for the convergence of* $G(\varphi_1, \varphi_2, \cdots)$.

COROLLARY 2. *If* $\Pi_{n=1}^{\infty} |\varphi_n|^2 = 0$, *then* $G(\varphi_1, \varphi_2, \cdots) = 0$.

UNIVERSITY OF CINCINNATI,
March, 1912.